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Extragradient Methods and Linesearch Algorithms for Solving Ky Fan Inequalities and Fixed Point Problems

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Abstract In this paper, we introduce some new iterative methods for finding a common element of the set of points satisfying a Ky Fan inequality, and the set of fixed points of a contraction mapping in a Hilbert space. The strong convergence of the iterates generated by each method is obtained thanks to a hybrid projection method, under the assumptions that the fixed-point mapping is a ξ-strict pseudocontraction, and the function associated with the Ky Fan inequality is pseudomonotone and weakly continuous. A Lipschitz-type condition is assumed to hold on this function when the basic iteration comes from the extragradient method. This assumption is unnecessary when an Armijo backtracking linesearch is incorporated in the extragradient method. The particular case of variational inequality problems is examined in a last section.

Keywords Ky Fan's inequality · Fixed-point problem · Hybrid projection method · Extragradient method · ξ-Strict pseudocontraction · Armijo backtracking linesearch · Lipschitz continuity

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1 Introduction

Fixed-point problems and variational inequalities are very useful and efficient tools in mathematics in the sense that they provide a unified framework for investigating problems arising in engineering sciences, structural analysis, and other fields; see, for instance, [1, 2]. A particular problem of interest is to find common elements of the set of fixed points of operators and the set of solutions of variational inequalities [3–9]. The motivation for studying such a problem is in its possible application to mathematical models whose constraints can be expressed as fixed-point problems and/or variational inequalities. This happens, in particular, in the practical problems as signal processing, network resource allocation, image recovery; see, for instance, [10–14]. The purpose of this work is to propose a strongly convergent method for finding a common element of the set of fixed points of a wide class of operators and the set of solutions of a variational problem associated with the Ky Fan inequality [15]. This latter problem is very general in the sense that it includes, as special cases, the optimization problem, the variational inequality, the saddle point problem, the Nash equilibrium problem in noncooperative games, the fixed-point problem, and others; see, for instance, [16–19] and references quoted therein. The interest of this problem is that it unifies all these particular problems in a convenient way. Let us also mention that this problem is often improperly called the “equilibrium problem”.

In most papers, the method used for solving the Ky Fan inequality problem is the proximal point method [20–25] which consists in solving at each iteration a nonlinear variational inequality problem. In this paper, we propose instead, to use an extragradient method with or without the incorporation of a linesearch. At each iteration, one or two convex minimization problems must be solved depending on the presence or not of a linesearch. Working in a Hilbert space, these methods usually generate sequences of iterates that only converge weakly to a solution of the problem. However, it is well known that strongly convergent algorithms are of fundamental importance for solving problems in infinite dimensional spaces [26]. In the literature, there exist two methods for obtaining the strong convergence from the weak convergence without additional assumptions on the data of the problem: the viscosity method [13, 14, 27–29] and the hybrid projection method [30, 31]. In the latter method, the solution set of the problem is outer approximated by a sequence of polyhedral subsets and the sequence of iterates converges to the orthogonal projection of a given point onto the solution set. In this paper, we incorporate the hybrid projection method into the extragradient method in order to obtain a strongly convergent sequence to a solution of the problem.

After recalling some preliminaries in Sect. 2, we combine in Sect. 3 the extragradient method and the hybrid projection method for obtaining a strong convergence algorithm under a Lipschitz-type condition on the function defining the Ky Fan inequality. In Sect. 4, we modify the previous algorithm in order to avoid the use of the Lipschitz-type property. The strategy is to replace the second step of the extragradient method by a linesearch followed by a projection. Several variants of the algorithm are obtained associated with different projections. Finally, in Sect. 5, we examine the particular case of variational inequality problems.

2 Preliminaries

Let C be a nonempty, closed, convex set in a real Hilbert space H and let f be a bifunction from $C \times C$ to $\mathbb{R}$. We consider the following problem:

$$\text{(EP)} \quad \text{Find } x^* \in C \text{ such that } f\big(x^*, y\big) \geq 0 \text{ for every } y \in C.$$

We refer to this problem as the Ky Fan inequality [15], and we denote the set of solutions of (EP) by $E(f)$. Simultaneously, we consider the fixed-point problem:

$$\text{Find } x^* \in C \text{ such that } Sx^* = x^*,$$

where S is a nonexpansive mapping from C to C. The set of fixed points of S is denoted by $\mathrm{Fix}(S)$.

When $f(x, y) = \langle F(x), y - x\rangle$ for every $x, y \in C$, where $F : C \to H$ is a continuous mapping, and $\langle \cdot, \cdot \rangle$ denotes the scalar product in H, the Ky Fan inequality problem reduces to a variational inequality problem, denoted (VIP). It consists in finding a point $x^* \in C$ such that

$$\big\langle F\big(x^*\big), y - x^*\big\rangle \geq 0 \quad \text{for every } y \in C.$$

The set of solutions of (VIP) is denoted by $\mathrm{VI}(F)$.

Most of the methods used in the literature for solving the fixed-point problem are derived from Mann's iteration algorithm [32], namely: Given $x_0 \in C$, compute, for all $n \in \mathbb{N}$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S x_n,$$

where the sequence $\{\alpha_n\}$ must satisfy some properties to force the weak convergence of the sequence $\{x_n\}$ to a fixed point of S.

On the other hand, many methods devoted to finding an element of $E(f)$ use the following algorithm: Given $x_0 \in C$, find, for all $n \in \mathbb{N}$, $x_{n+1} \in C$ such that

$$f(x_{n+1}, y) + \frac{1}{r_n}\langle y - x_{n+1}, x_{n+1} - x_n\rangle \geq 0 \quad \text{for every } y \in C, \tag{1}$$

where the sequence $\{r_n\}$ is positive. The combination of these two methods gives rise to the following algorithm for finding an element of $E(f) \cap \mathrm{Fix}(S)$: Given $x_0 \in H$, compute

$$\begin{cases} z_n \in C \text{ such that } f(z_n, y) + \frac{1}{r_n}\langle y - z_n, z_n - x_n\rangle \geq 0 \quad \text{for every } y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S z_n, \end{cases}$$

for all $n \in \mathbb{N}$. The sequence $\{x_n\}$ generated by this scheme converges weakly to some $x^* \in E(f) \cap \mathrm{Fix}(S)$, provided that $\{\alpha_n\} \subset [a, b]$ for some $a, b \in]0, 1[$, and $\{r_n\} \subset]0, \infty[$ satisfy $\liminf r_n > 0$, [25, Theorem 4.1].

In this paper, we propose to consider an extragradient-type iteration instead of the proximal iteration used above. Introduced by Korpelevich [33] for solving variational inequalities, the extragradient method uses two projections per iteration:

Given $x_n \in C$, compute

$$y_n = P_C\big(x_n - \lambda_n F(x_n)\big) \quad \text{and} \quad x_{n+1} = P_C\big(x_n - \lambda_n F(y_n)\big), \tag{2}$$

where P_C denotes the orthogonal projection onto C. Let us recall that this method is convergent when F is pseudomonotone and Lipschitz continuous. Many papers introduce the extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for a monotone Lipschitz continuous mapping [3–9].

Recently, the extragradient method has been generalized by Tran et al. [34] for solving the Ky Fan inequality in $\mathbb{R}^n$. In that case, the two steps (2) become: Given $x_n \in C$, find successively y_n and x_{n+1} as follows:

$$\begin{cases} y_n = \arg\min_{y\in C}\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ x_{n+1} = \arg\min_{y\in C}\{\lambda_n f(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \end{cases}$$

where $\{\lambda_n\} \subset]0, 1]$. This method has been proven convergent to some $x^* \in E(f)$ when f is pseudomonotone and satisfies a Lipschitz-type property. However, this latter condition is very strong and difficult to check. So, in [34], the authors replaced the computation of x_{n+1} by an Armijo backtracking linesearch, followed by a projection onto an hyperplane. More precisely, given $x_n \in C$, the iterates y_n, z_n, and x_{n+1} are calculated as follows:

$$\begin{cases} y_n = \arg\min_{y\in C}\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ z_n = (1-\gamma^m)x_n + \gamma^m y_n, \text{ where } m \text{ is the smallest nonnegative integer} \\ \quad \text{such that } f(z_n, x_n) - f(z_n, y_n) \geq \frac{\alpha}{2\lambda_n}\|x_n - y_n\|^2, \\ x_{n+1} = P_C(x_n - \sigma_n g_n) \text{ with } g_n \in \partial_2 f(z_n, x_n) = \partial[f(z_n, \cdot)](x_n) \text{ and} \\ \sigma_n = f(z_n, x_n)/\|g_n\|^2, \end{cases}$$

where the parameters α, γ, α_n, and λ_n are chosen as explained in [34]. Doing so, the convergence is obtained without a Lipschitz-type condition.

Our aim, in this paper, is first to combine the extragradient iteration (with or without a linesearch) and the fixed-point iteration, and after to use a hybrid projection method to obtain a sequence $\{x_n\}$ that converges strongly to a point $x^* \in E(f) \cap \mathrm{Fix}(S)$.

To end this section, we give some results that are generalizations, in infinite dimensional spaces, of Theorem 10.8 and Corollary 10.8.1 in [35]. They will be used in Sect. 4.

Proposition 2.1 *Let H be a real Hilbert space. Consider $\{\varphi_n\}$ a sequence of continuous convex functions from H into $\mathbb{R}$, and φ a continuous convex function from H into $\mathbb{R}$. If $\{\varphi_n\}$ converges pointwise to φ on H, then there exists $\eta > 0$ such that $\{\varphi_n\}$ converges uniformly to φ on ηB where B denotes the unit closed ball in H.*

Proof Since, by assumption, the sequence $\{\varphi_n(x)\}$ is bounded for every $x \in H$, it follows from Theorem 2.2.22 in [36] that the sequence $\{\varphi_n\}$ is locally equi-Lipschitz on H, and thus also locally equi-bounded on H. So, there exist $\delta > 0$ and $M > 0$ such that, for every $x \in \delta B$ and $n \in \mathbb{N}$, we have $\varphi_n(x) \leq M$.

Now, let $\mathcal{S}$ be the collection of singletons in H. Since the sequence $\{\varphi_n\}$ converges pointwise to φ, it also $\mathcal{S}$-converges to φ (see Lemma 1.4 in [37]). Then the assumptions of Lemma 1.5 in [37] are satisfied with $W = \{0\}$, and consequently $\{\varphi_n\}$ converges uniformly to φ on ηB for some $0 < \eta < \delta$. □

Corollary 2.1 *Let $\{\varphi_n\}$ be a sequence of continuous convex functions from H into $\mathbb{R}$, and let φ be a continuous convex function from H into $\mathbb{R}$ such that*

$$\limsup_{n\to\infty} \varphi_n(x) \le \varphi(x) \quad \forall x \in H.$$

Then there exists $\eta > 0$ such that, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\varphi_n(x) \le \varphi(x) + \varepsilon \quad \forall n \ge n_0, \quad \forall x \in \eta B,$$

where B denotes the unit closed ball in H.

Proof Let $\psi_n = \max\{\varphi_n, \varphi\}$. Then the sequence $\{\psi_n\}$ of convex continuous functions converges pointwise to φ on H. So, by Proposition 2.1, there exists $\eta > 0$ such that $\{\psi_n\}$ converges uniformly to φ on ηB. Hence, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\left|\psi_n(x) - \varphi(x)\right| < \varepsilon \quad \forall n \ge n_0, \quad \forall x \in \eta B.$$

Since $\varphi_n \le \psi_n$ and $\psi_n(x) - \varphi(x) \ge 0$, we obtain the desired result

$$\varphi_n(x) \le \psi_n(x) < \varphi(x) + \varepsilon \quad \forall n \ge n_0, \quad \forall x \in \eta B.$$ □

3 An Extragradient Algorithm

In this section, first we combine the extragradient method for solving (EP) with a fixed-point method. More precisely, we consider the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{t_n\}$ generated by $x_0 \in C$ and

$$\begin{cases} y_n = \arg\min_{y\in C}\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ z_n = \arg\min_{y\in C}\{\lambda_n f(y_n, y) + \frac{1}{2}\|y - x_n\|^2\}, \\ t_n = \alpha_n x_n + (1 - \alpha_n)[\beta_n z_n + (1 - \beta_n)Sz_n], \\ x_{n+1} = t_n, \end{cases}$$

for every $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0, 1[$, $\{\beta_n\} \subset]0, 1[$, and $\{\lambda_n\} \subset]0, 1]$.

Here, we assume that the following conditions are satisfied on the bifunction $f : C \times C \to \mathbb{R}$.

(A1) $f(x, x) = 0$ for every $x \in C$;
(A2) f is pseudomonotone on C, i.e., $f(x, y) \ge 0 \Rightarrow f(y, x) \le 0\ \forall x, y \in C$;
(A3) f is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in C converging weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$;
(A4) $f(x, \cdot)$ is convex, lower semicontinuous, and subdifferentiable on C for every $x \in C$;

(A5) f satisfies the Lipschitz-type condition: $\exists c_1 > 0, \exists c_2 > 0$, such that for every $x, y, z \in C$,

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|y-x\|^2 - c_2\|z-y\|^2.$$

It is easy to see that if f satisfies the properties (A1)–(A4), then the set $E(f)$ of solutions to the Ky Fan inequality is closed and convex.

Remark 3.1 A first example of function f satisfying assumption (A5), is given by

$$f(x,y) = \langle F(x), y-x \rangle \quad \text{for every } x, y \in C,$$

where $F : C \to H$ is Lipschitz continuous on C (with constant $L > 0$) [18]. In that example, $c_1 = c_2 = L/2$. Another example, related to the Cournot–Nash equilibrium model, is described in [34, p. 768]. The function $f : C \times C \to \mathbb{R}$ is defined, for every $x, y \in C$, by

$$f(x,y) = \langle F(x) + Qy + q, y-x \rangle,$$

with $C = \{x \in \mathbb{R}^n : Ax \leq b\}$, $F : C \to \mathbb{R}^n$, $Q \in \mathbb{R}^{n\times n}$, a symmetric positive semidefinite matrix, and $q \in \mathbb{R}^n$. If F is Lipschitz continuous on C (with constant $L > 0$), then f satisfies (A5) with $c_1 > 0$, $c_2 > 0$ such that $2\sqrt{c_1 c_2} \geq L + \|Q\|$.

Furthermore, we also suppose that the mapping S satisfies the condition:

(B1) S is a ξ-strict pseudocontraction mapping for some $\xi \in [0, 1[$.

Let us recall that S is a ξ-strict pseudocontraction mapping iff there exists a scalar $\xi \in [0, 1[$ such that

$$\|Sx - Sy\|^2 \leq \|x-y\|^2 + \xi \big\|(x - Sx) - (y - Sy)\big\|^2 \quad \text{for every } x, y \in C.$$

It is easy to see that a nonexpansive mapping on C is also a 0-strict pseudocontraction mapping. Furthermore [30], if S is a ξ-strict pseudocontraction mapping, then the fixed-point set Fix(S) is closed and convex and the mapping $I - S$ is demiclosed at zero, i.e., satisfies the property

$$x_n \rightharpoonup x \text{ (weakly)} \quad \text{and} \quad Sx_n - x_n \to 0 \text{ (strongly)} \quad \Rightarrow \quad Sx = x.$$

In the sequel, we also suppose that the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ satisfy the conditions

$$\text{(P)} \qquad \begin{cases} \{\alpha_n\} \subset [0, c] \text{ for some } c < 1, \{\beta_n\} \subset [d, b] \text{ for some } 0 \leq \xi < d \leq b < 1, \\ \{\lambda_n\} \subset [\lambda_{\min}, \lambda_{\max}], \text{ where } 0 < \lambda_{\min} \leq \lambda_{\max} < \min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}. \end{cases}$$

Now, let $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, and $\{t_n\}$ be the sequences generated by the combination of the extragradient method and the fixed-point method described at the beginning of this section. These sequences satisfy the following properties:

Proposition 3.1 [27, Lemma 3.1] *For every $x^* \in E(f)$, and every $n \in \mathbb{N}$, one has*

(i) $\langle x_n - y_n, y - y_n \rangle \leq \lambda_n f(x_n, y) - \lambda_n f(x_n, y_n)\ \forall y \in C$,
(ii) $\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - 2\lambda_n c_1)\|y_n - x_n\|^2 - (1 - 2\lambda_n c_2)\|z_n - y_n\|^2$.

Proposition 3.2 *For every $x^* \in E(f) \cap \mathrm{Fix}(S)$, and every $n \in \mathbb{N}$, one has*

$$\begin{aligned}\|t_n - x^*\|^2 \le \|x_n - x^*\|^2 &- (1-\alpha_n)(1-2\lambda_n c_1)\|y_n - x_n\|^2 \\ &- (1-\alpha_n)(1-2\lambda_n c_2)\|z_n - y_n\|^2 \\ &- (1-\alpha_n)(1-\beta_n)(\beta_n - \xi)\|z_n - Sz_n\|^2.\end{aligned}$$

Proof Let $x^* \in E(f) \cap \mathrm{Fix}(S)$. Using the convexity of $\|\cdot\|^2$ and the equality

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2,$$

valid for any $t \in [0,1]$ and for any $x, y \in H$, we obtain successively

$$\begin{aligned}\|t_n - x^*\|^2 &= \|\alpha_n(x_n - x^*) + (1-\alpha_n)[\beta_n z_n + (1-\beta_n)Sz_n - x^*]\|^2 \\ &\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|\beta_n z_n + (1-\beta_n)Sz_n - x^*\|^2 \\ &= \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\big[\beta_n\|z_n - x^*\|^2 + (1-\beta_n)\|Sz_n - x^*\|^2 \\ &\quad - \beta_n(1-\beta_n)\|Sz_n - z_n\|^2\big] \\ &\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\big[\beta_n\|z_n - x^*\|^2 + (1-\beta_n)\|z_n - x^*\|^2 \\ &\quad + (1-\beta_n)\xi\|z_n - Sz_n\|^2 - \beta_n(1-\beta_n)\|Sz_n - z_n\|^2\big] \\ &= \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2 \\ &\quad - (1-\alpha_n)(1-\beta_n)(\beta_n - \xi)\|Sz_n - z_n\|^2, \end{aligned} \tag{3}$$

where we have used the ξ-strict pseudo-contraction property of the mapping S to get the last inequality. Finally, it remains to apply Proposition 3.1(ii) to obtain the announced result. □

In order to obtain the strong convergence of the sequence $\{x_n\}$ generated by the combination of the extragradient method and the fixed-point method, we can use the following result.

Proposition 3.3 [30] *Let K be a nonempty closed convex subset of H. Let $u \in H$ and let $\{x_n\}$ be a sequence in H. If any weak limit point of $\{x_n\}$ belongs to K, and $\|x_n - u\| \le \|u - P_K u\|$ for all $n \in \mathbb{N}$, then $x_n \to P_K u$.*

In order to apply Proposition 3.3 in our context, we set $K = E(f) \cap \mathrm{Fix}(S)$ and $u = x_0$. Furthermore, we impose that the sequence $\{x_n\}$ generated by our algorithms satisfies, for all $n \in \mathbb{N}$, the inequality

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\| \le \|\tilde{x}_0 - x_0\|, \tag{4}$$

where $\tilde{x}_0 = P_{E(f)\cap \mathrm{Fix}(S)} x_0$. In that case, the sequence $\{\|x_n - x_0\|\}$ is convergent and the sequence $\{x_n\}$ is bounded. These properties will be useful to prove that any weak limit point of $\{x_n\}$ belongs to $E(f) \cap \mathrm{Fix}(S)$.

In order to construct a sequence $\{x_n\}$ satisfying (4), we consider an outer approximation method, that is, we construct a sequence $\{\Omega_n\}$ of subsets of C such that

$$\Omega_n \supset E(f) \cap \operatorname{Fix}(S) \quad \forall n \in \mathbb{N} \quad \text{and} \quad P_{\Omega_n} x_0 \to \tilde{x}_0 \quad \text{as } n \to \infty,$$

and we set $x_{n+1} = P_{\Omega_n} x_0$. In that purpose, from the equality

$$\|x_{n+1} - x_0\|^2 = \|x_{n+1} - x_n\|^2 + \|x_n - x_0\|^2 + 2\langle x_{n+1} - x_n, x_n - x_0\rangle, \qquad (5)$$

we observe that the first inequality of (4) is satisfied when $\langle x_{n+1} - x_n, x_n - x_0\rangle \geq 0$, i.e., when $x_{n+1} \in D_n$, where

$$D_n = \big\{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\big\}.$$

On the other hand, when the sequences of parameters satisfy (P), it follows from Proposition 3.2 that

$$C_n = \big\{z \in C : \|t_n - z\| \leq \|x_n - z\|\big\} \supset E(f) \cap \operatorname{Fix}(S).$$

Consequently, if we set $\Omega_n = C_n \cap D_n$, and prove that $C_n \cap D_n \supset E(f) \cap \operatorname{Fix}(S)$, then $x_{n+1} = P_{C_n \cap D_n} x_0$ will satisfy the inequalities (4). It is the aim of the next proposition.

Proposition 3.4 *For every $n \in \mathbb{N}$, the sets C_n and D_n defined above are closed and convex. Furthermore, when $E(f) \cap \operatorname{Fix}(S) \neq \emptyset$ and $x_{n+1} = P_{C_n \cap D_n} x_0$, one has*

$$E(f) \cap \operatorname{Fix}(S) \subset C_n \cap D_n.$$

Proof Obviously, C_n and D_n are closed and D_n is convex for every $n \in \mathbb{N}$. Since we can write C_n under the form

$$C_n = \big\{z \in C : \|t_n - x_n\|^2 + 2\langle t_n - x_n, x_n - z\rangle \leq 0\big\},$$

we see immediately that C_n is also convex for every $n \in \mathbb{N}$. Furthermore, by Proposition 3.2, we have that $E(f) \cap \operatorname{Fix}(S) \subset C_n$ for every $n \in \mathbb{N}$.

Next, we prove, by induction on n, that $E(f) \cap \operatorname{Fix}(S) \subset C_n \cap D_n$.

For $n = 0$, we have $D_0 = C$ and $E(f) \cap \operatorname{Fix}(S) \subset C_0 \cap D_0$. Now, suppose that for some $n \in \mathbb{N}$, $E(f) \cap \operatorname{Fix}(S) \subset C_n \cap D_n$. Since $E(f) \cap \operatorname{Fix}(S)$ is nonempty, $C_n \cap D_n$ is a nonempty closed convex subset of C. So, $x_{n+1} = P_{C_n \cap D_n} x_0$ is well defined, and $\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0$ holds for every $z \in C_n \cap D_n$. Since $E(f) \cap \operatorname{Fix}(S) \subset C_n \cap D_n$, we have, in particular, $\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \geq 0$, for every $z \in E(f) \cap \operatorname{Fix}(S)$, and consequently, $E(f) \cap F(S) \subset D_{n+1}$. Therefore, we obtain $E(f) \cap \operatorname{Fix}(S) \subset C_{n+1} \cap D_{n+1}$. □

The inequalities (4) being satisfied, the sequence $\{x_n\}$ defined by $x_{n+1} = P_{C_n \cap D_n} x_0$, for every $n \in \mathbb{N}$, is also bounded, and to obtain the strong convergence of that sequence to the projection of x_0 onto $E(f) \cap \operatorname{Fix}(S)$, it is sufficient, thanks to Proposition 3.4, to prove that every weak limit point of $\{x_n\}$ belongs to $E(f) \cap \operatorname{Fix}(S)$. This method is known in the literature as the hybrid projection method [30]. It is that method we will use in each of our algorithms to get the strong convergence of the iterates. Combining the hybrid projection method described above, with the extragradient method, we obtain the following basic algorithm.

Algorithm 1

Step 0. Choose the sequences $\{\alpha_n\} \subset [0, 1[$, $\{\beta_n\} \subset]0, 1[$, and $\{\lambda_n\} \subset]0, 1]$.
Step 1. Let $x_0 \in C$. Set $n = 0$.
Step 2. Solve successively the strongly convex programs

$$\min_{y \in C} \left\{ \lambda_n f(x_n, y) + \frac{1}{2} \|y - x_n\|^2 \right\} \quad \text{and}$$

$$\min_{y \in C} \left\{ \lambda_n f(y_n, y) + \frac{1}{2} \|y - x_n\|^2 \right\}$$

to obtain the unique optimal solutions y_n and z_n, respectively.
Step 3. Compute $t_n = \alpha_n x_n + (1 - \alpha_n)[\beta_n z_n + (1 - \beta_n) S z_n]$.
If $y_n = x_n$ and $t_n = x_n$, then STOP: $x_n \in E(f) \cap \mathrm{Fix}(S)$.
Otherwise, go to Step 4.
Step 4. Compute $x_{n+1} = P_{C_n \cap D_n} x_0$, where

$$C_n = \left\{ z \in C : \|t_n - z\| \leq \|x_n - z\| \right\} \quad \text{and}$$

$$D_n = \left\{ z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0 \right\}.$$

Step 5. Set $n := n + 1$, and go to Step 2.

Before proving the strong convergence of the iterates generated by Algorithm 1, we justify the stopping criterion in the next proposition.

Proposition 3.5 *If $y_n = x_n$, then $x_n \in E(f)$. If $y_n = x_n$ and $t_n = x_n$, then $x_n \in E(f) \cap \mathrm{Fix}(S)$.*

Proof When $y_n = x_n$, it follows from Proposition 2.1(i) that

$$0 \leq \lambda_n f(x_n, y) - \lambda_n f(x_n, x_n) = \lambda_n f(x_n, y)$$

holds, for every $y \in C$. But this means that $x_n \in E(f)$.

On the other hand, when $y_n = x_n$ and $t_n = x_n$, we have that $z_n = x_n$, and thus

$$x_n = \alpha_n x_n + (1 - \alpha_n)\left[\beta_n x_n + (1 - \beta_n) S x_n\right].$$

Since, by assumption, $1 - \alpha_n > 0$ and $1 - \beta_n > 0$, it follows that $x_n = S x_n$. So, $x_n \in \mathrm{Fix}(S)$. □

Now we suppose that the STOP, in Step 3, never occurs in Algorithm 1 and we prove the strong convergence of the infinite sequence $\{x_n\}$ generated by this algorithm to the projection of x_0 onto $E(f) \cap \mathrm{Fix}(S)$.

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of H. Let f be a function from $\Delta \times \Delta$ into $\mathbb{R}$ satisfying conditions* (A1)–(A5), *and let S be a mapping from C to C satisfying conditions* (B1), *and such that $E(f) \cap \mathrm{Fix}(S) \neq \emptyset$. Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ satisfy the conditions* (P). *Then the sequence*

$\{x_n\}$ generated by Algorithm 1 converges strongly to the projection of x_0 onto the set $E(f) \cap \mathrm{Fix}(S)$.

Proof Let $\{x_n\}$ be the infinite sequence generated by Algorithm 1. Since we use the hybrid projection method, it follows from our previous discussion that the sequence $\{x_n\}$ generated by Algorithm 1 satisfies the inequalities (4), and thus is bounded. Consequently, from Proposition 3.3, the strong convergence of the sequence $\{x_n\}$ to the projection of x_0 onto $\mathrm{Fix}(S) \cap E(f)$ holds, provided that every weak limit point of $\{x_n\}$ is an element of $E(f) \cap \mathrm{Fix}(S)$. So, let $\bar{x}$ be a weak limit point of $\{x_n\}$, and suppose that $x_{n_i} \rightharpoonup \bar{x}$. Since C is closed and convex, it is also weakly closed, and thus $\bar{x} \in C$. The proof of $\bar{x} \in E(f) \cap \mathrm{Fix}(S)$ is done in several steps.

Step 1. $\|x_{n+1} - x_n\| \to 0$, $\|x_n - t_n\| \to 0$, $\|x_n - y_n\| \to 0$, $\|x_n - z_n\| \to 0$, $\|Sz_n - z_n\| \to 0$.

The sequence $\{\|x_n - x_0\|\}$, being nondecreasing and bounded thanks to (4), is convergent to some $a \geq 0$. But then, since $x_{n+1} \in D_n$, it follows from (5) that, for every $n \in \mathbb{N}$,

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \tag{6}$$

and thus that $\|x_{n+1} - x_n\| \to 0$, because the right-hand side of (6) tends to zero.

Now, $\|t_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$ because $x_{n+1} \in C_n$. Hence,

$$\|x_n - t_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - t_n\| \leq 2\|x_n - x_{n+1}\|.$$

Since $\|x_n - x_{n+1}\| \to 0$, we deduce that $\|x_n - t_n\| \to 0$.

Next, let $x^* \in E(f) \cap \mathrm{Fix}(S)$. Then, using Proposition 3.2, we can write, for every $n \in \mathbb{N}$, that

$$(1-\alpha_n)(1-2\lambda_n c_1)\|y_n - x_n\|^2 \leq \big[\|x_n - x^*\| + \|t_n - x^*\|\big]\|x_n - t_n\|,$$

$$(1-\alpha_n)(1-2\lambda_n c_2)\|z_n - y_n\|^2 \leq \big[\|x_n - x^*\| + \|t_n - x^*\|\big]\|x_n - t_n\|,$$

$$(1-\alpha_n)(1-\beta_n)(\beta_n - \xi)\|Sz_n - z_n\|^2 \leq \big[\|x_n - x^*\| + \|t_n - x^*\|\big]\|x_n - t_n\|,$$

because $\|x_n - x^*\| - \|t_n - x^*\| \leq \|x_n - t_n\|$.

Since, for every $n \in \mathbb{N}$,

$$1-\alpha_n \geq 1-c > 0, \qquad 1-2\lambda_n c_1 \geq 1-2\lambda_{\max} c_1 > 0, \qquad \|x_n - t_n\| \to 0,$$

$$1-2\lambda_n c_2 \geq 1-2\lambda_{\max} c_2 > 0, \qquad 1-\beta_n \geq 1-b > 0, \qquad \beta_n - \xi \geq d - \xi > 0,$$

and since the sequences $\{x_n\}$ and $\{t_n\}$ are bounded, it follows that

$$\|y_n - x_n\| \to 0, \qquad \|z_n - y_n\| \to 0, \qquad \|z_n - x_n\| \to 0, \quad \text{and} \quad \|Sz_n - z_n\| \to 0.$$

Step 2. $\bar{x} \in E(f)$.

Since $x_{n_i} \rightharpoonup \bar{x}$ and $\|x_n - y_n\| \to 0$, we have that $y_{n_i} \rightharpoonup \bar{x}$. On the other hand, using Proposition 3.1(i), we have, for every $y \in C$, and for every $i \in \mathbb{N}$, that

$$\langle x_{n_i} - y_{n_i}, y - y_{n_i}\rangle \leq \lambda_{n_i} f(x_{n_i}, y) - \lambda_{n_i} f(x_{n_i}, y_{n_i}). \tag{7}$$

Since $\|x_{n_i} - y_{n_i}\| \to 0$ and $y - y_{n_i} \rightharpoonup y - \bar{x}$ as $i \to \infty$, and since, for every $i \in \mathbb{N}$, $0 < \lambda_{\min} \le \lambda_{n_i} \le \lambda_{\max}$, we obtain, after taking the limit in (7), that

$$f(\bar{x}, y) \ge 0 \quad \text{for every } y \in C,$$

i.e., $\bar{x} \in E(f)$.

Step 3. $\bar{x} \in \mathrm{Fix}(S)$.

Since $x_{n_i} \rightharpoonup \bar{x}$ and $\|x_n - z_n\| \to 0$, we have that $z_{n_i} \rightharpoonup \bar{x}$. On the other hand, we know that $\|Sz_{n_i} - z_{n_i}\| \to 0$. Consequently, the mapping $I - S$ being demiclosed at zero, it follows immediately that $S\bar{x} - \bar{x} = 0$, i.e., $\bar{x} \in \mathrm{Fix}(S)$. □

4 Extragradient Algorithms with Linesearches

The extragradient-type algorithm considered in the previous section has been proven to be strongly convergent under the Lipschitz-type condition (A5). This condition depends on two positive parameters c_1 and c_2 and, in some cases, they are unknown or difficult to approximate. In this section, we modify the second step of the extragradient iteration, i.e., the computation of z_n, by including a linesearch and a projection onto a hyperplane. Considering two different projections, we obtain two algorithms (Algorithms 2 and 2a) whose strong convergence can be established without assuming the Lipschitz-type condition (A5). However, in compensation, we have to slightly reinforce assumption (A3). More precisely, in this section, we suppose that the function f satisfies conditions (A1), (A2), (A4), as well as the following condition:

(A3bis) f is jointly weakly continuous on the product $\Delta \times \Delta$ where Δ is an open convex set containing C in the sense that if $x, y \in \Delta$ and $\{x_n\}$ and $\{y_n\}$ are two sequences in Δ converging weakly to x and y, respectively, then $f(x_n, y_n) \to f(x, y)$.

Furthermore, we also suppose that the mapping S satisfies condition (B1) and the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ satisfy the conditions:

$$(\mathrm{Q}) \quad \begin{cases} \{\alpha_n\} \subset [0, c] \text{ for some } c < 1, \{\beta_n\} \subset [d, b] \text{ for some } 0 \le \xi < d \le b < 1, \\ \{\lambda_n\} \subset [\lambda, 1] \text{ for some } 0 < \lambda \le 1. \end{cases}$$

Under these assumptions, we will obtain an algorithm strongly converging to a point $x^* \in E(f) \cap \mathrm{Fix}(S)$ under the same assumptions as the ones used by the proximal-type methods [20–25]. However, our subproblems, being strongly convex minimization problems, seem easier to solve than the proximal subproblem (1).

Our first extragradient-type algorithm with a linesearch can be expressed as follows.

Algorithm 2

Step 0. Choose $\alpha \in]0, 2[$, $\gamma \in]0, 1[$ and the sequences

$$\{\alpha_n\} \subset [0, 1[, \qquad \{\beta_n\} \subset]0, 1[\quad \text{and} \quad \{\lambda_n\} \subset]0, 1].$$

Step 1. Let $x_0 \in C$. Set $n = 0$.

Step 2. Solve the strongly convex program $\min_{y\in C}\{\lambda_n f(x_n,y)+\frac{1}{2}\|y-x_n\|^2\}$ to obtain the unique optimal solution y_n.

Step 3. If $y_n=x_n$, then set $z_n=x_n$. Otherwise

Step 3.1. Find m the smallest nonnegative integer such that

$$\begin{cases} f(z_{n,m},x_n)-f(z_{n,m},y_n)\geq \frac{\alpha}{2\lambda_n}\|x_n-y_n\|^2 \text{ where} \\ z_{n,m}=(1-\gamma^m)x_n+\gamma^m y_n. \end{cases}$$

Step 3.2. Set $\rho_n=\gamma^m$, $z_n=z_{n,m}$, and go to Step 4.

Step 4. Select $g_n\in\partial_2 f(z_n,x_n)$ and compute $w_n=P_C(x_n-\sigma_n g_n)$ where $\sigma_n=\frac{f(z_n,x_n)}{\|g_n\|^2}$ if $y_n\neq x_n$ and $\sigma_n=0$ otherwise.

Step 5. Compute $t_n=\alpha_n x_n+(1-\alpha_n)[\beta_n w_n+(1-\beta_n)Sw_n]$.
If $y_n=x_n$ and $t_n=x_n$, then STOP: $x_n\in E(f)\cap \mathrm{Fix}(S)$.
Otherwise, go to Step 6.

Step 6. Compute $x_{n+1}=P_{C_n\cap D_n}x_0$, where

$$C_n=\{z\in C:\|t_n-z\|\leq\|x_n-z\|\} \quad \text{and}$$

$$D_n=\{z\in C:\langle x_n-z,x_0-x_n\rangle\geq 0\}.$$

Step 7. Set $n:=n+1$, and go to Step 2.

First we prove that the linesearch is well defined and that the steplength σ_n is positive when $y_n\neq x_n$ (see Steps 3–4 of Algorithm 2).

Proposition 4.1 *Suppose that $y_n\neq x_n$ for some $n\in\mathbb{N}$. Then*

(i) *The linesearch corresponding to x_n and y_n (Step 3.1) is well defined.*
(ii) $f(z_n,x_n)>0$.
(iii) $0\notin\partial_2 f(z_n,x_n)$.

Proof We only prove the first statement. The proof of the other ones can be found in [34, Lemma 4.5]. Let $n\in\mathbb{N}$ and suppose, to get a contradiction that the following inequality hold for every integer $m\geq 0$

$$f(z_{n,m},x_n)-f(z_{n,m},y_n)<\frac{\alpha}{2\lambda_n}\|x_n-y_n\|^2, \tag{8}$$

where $z_{n,m}=(1-\gamma^m)x_n+\gamma^m y_n$ and $\gamma\in]0,1[$. Then $\{z_{n,m}\}_m$ converges strongly to x_n as $m\to\infty$, and thus also weakly. Since $f(\cdot,x)$ is weakly continuous on an open set $\Delta\supset C$ for every $x\in\Delta$, it follows that $f(z_{n,m},x_n)\to f(x_n,x_n)=0$, and that $f(z_{n,m},y_n)\to f(x_n,y_n)$. So, taking the limit on m in (8), yields

$$-f(x_n,y_n)\leq\frac{\alpha}{2\lambda_n}\|y_n-x_n\|^2. \tag{9}$$

Now, from Proposition 3.1(i) with $y=x_n$, we can write that

$$\|y_n-x_n\|^2\leq-\lambda_n f(x_n,y_n).$$

Combining this inequality with (9), we obtain that $(1-\frac{\alpha}{2})\|y_n - x_n\|^2 \le 0$. Since $\alpha \in]0,2[$, we deduce that $y_n = x_n$, which gives rise to a contradiction, because we have supposed that $y_n \ne x_n$. Consequently, the linesearch is well defined. □

In the next proposition, we justify the stopping criterion.

Proposition 4.2 *If $y_n = x_n$, then $x_n \in E(f)$. If $y_n = x_n$ and $t_n = x_n$, then $w_n = x_n$ and $x_n \in E(f) \cap \mathrm{Fix}(S)$.*

Proof When $y_n = x_n$, it follows from Proposition 3.1(i), that

$$0 \le \lambda_n f(x_n, y) - \lambda_n f(x_n, x_n) = \lambda_n f(x_n, y)$$

holds for every $y \in C$. This means that $x_n \in E(f)$.

On the other hand, when $y_n = x_n$ and $t_n = x_n$, we have that $w_n = x_n$, because $\sigma_n = 0$ and $x_n \in C$. Hence,

$$x_n = \alpha_n x_n + (1-\alpha_n)\big[\beta_n x_n + (1-\beta_n)Sx_n\big].$$

Since, by assumption, $1-\alpha_n > 0$ and $1-\beta_n > 0$, it follows that $Sx_n = x_n$, i.e., $x_n \in \mathrm{Fix}(S)$. □

The next proposition will be used in the proof of the convergence theorem. It is an infinite-dimensional version of Theorem 24.5 in [35].

Proposition 4.3 *Let $f : \Delta \times \Delta \to \mathbb{R}$ be a function satisfying conditions* (A3bis) *and* (A4). *Let $\bar{x}, \bar{z} \in \Delta$ and $\{x_n\}$ and $\{z_n\}$ be two sequences in Δ converging weakly to $\bar{x}, \bar{z}$, respectively. Then, for any $\varepsilon > 0$, there exist $\eta > 0$ and $n_\varepsilon \in \mathbb{N}$ such that*

$$\partial_2 f(z_n, x_n) \subset \partial_2 f(\bar{z}, \bar{x}) + \frac{\varepsilon}{\eta} B,$$

for every $n \ge n_\varepsilon$, where B denotes the closed unit ball in H.

Proof For every $n \in \mathbb{N}$, consider the convex functions $h_n := f(z_n, \cdot)$ and $h := f(\bar{z}, \cdot)$. Let also $d \in H$ and $\mu > h'(\bar{x}; d)$. Then, by definition of the directional derivative, there exists $\lambda > 0$ such that

$$\frac{h(\bar{x} + \lambda d) - h(\bar{x})}{\lambda} < \mu.$$

Since f is jointly weakly continuous on $\Delta \times \Delta$ and since $x_n + \lambda d \rightharpoonup \bar{x} + \lambda d$ and $x_n \rightharpoonup \bar{x}$, we obtain that $h_n(x_n + \lambda d) \to h(\bar{x} + \lambda d)$ and $h_n(x_n) \to h(\bar{x})$. Hence, for n sufficiently large, we have

$$\frac{h_n(x_n + \lambda d) - h_n(x_n)}{\lambda} < \mu. \tag{10}$$

Since $h_n'(x_n; d) \le \frac{h_n(x_n+\lambda d)-h_n(x_n)}{\lambda}$, and as (10) is true for any $\mu > h'(\bar{x}; d)$, it follows that

$$\limsup_{n\to\infty} h_n'(x_n; d) \le h'(\bar{x}; d) \quad \forall d \in H.$$

Then, from Corollary 2.1, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$h_n'(x_n; d) \le h'(\bar{x}; d) + \varepsilon \quad \forall n \ge n_0, \quad \forall d \in \eta B.$$

By positive homogeneity, we have

$$h_n'(x_n; d) \le h'(\bar{x}; d) + \frac{\varepsilon}{\eta}\|d\| \quad \forall n \ge n_0, \quad \forall d \in H.$$

Since the directional derivative is the support function of the subdifferential, we easily deduce that

$$\partial h_n(x_n) \subset \partial h(\bar{x}) + \frac{\varepsilon}{\eta} B.$$

However, this means that

$$\partial_2 f(z_n, x_n) \subset \partial_2 f(\bar{z}, \bar{x}) + \frac{\varepsilon}{\eta} B. \qquad \square$$

The next proposition will play the same role as Propositions 3.1 and 3.2, but for Algorithm 2.

Proposition 4.4 *For every $x^* \in E(f) \cap \mathrm{Fix}(S)$ and all $n \in \mathbb{N}$, one has*

(i) $\|w_n - x^*\|^2 \le \|x_n - x^*\|^2 - \sigma_n^2 \|g_n\|^2$,
(ii) $\|t_n - x^*\|^2 \le \|x_n - x^*\|^2 - (1-\alpha_n)\sigma_n^2\|g_n\|^2 - (1-\alpha_n)(1-\beta_n)(\beta_n - \xi) \times \|Sw_n - w_n\|^2$.

Proof (i) Using the definition of w_n, and the nonexpansiveness of the projection, we obtain

$$\begin{aligned}\|w_n - x^*\|^2 &\le \|x_n - \sigma_n g_n - x^*\|^2 \\ &= \|x_n - x^*\|^2 - 2\sigma_n\langle g_n, x_n - x^*\rangle + \sigma_n^2\|g_n\|^2. \qquad (11)\end{aligned}$$

On the other hand, using successively the definition of the subgradient g_n, the pseudomonotonicity of f on C, and the definition of σ_n, we can write that

$$\langle g_n, x_n - x^*\rangle \ge f(z_n, x_n) - f(z_n, x^*) \ge f(z_n, x_n) = \sigma_n\|g_n\|^2. \qquad (12)$$

Let us point out that this property also holds when $y_n = x_n$ because, in that case, $z_n = x_n$ and $\sigma_n = 0$. Combining (11) and (12), we easily deduce (i).

(ii) Let $x^* \in E(f) \cap \mathrm{Fix}(S)$. Then the inequality (3) can be easily obtained with w_n instead of z_n:

$$\begin{aligned}\|t_n - x^*\|^2 \le{}& \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|w_n - x^*\|^2 \\ &- (1-\alpha_n)(1-\beta_n)(\beta_n - \xi)\|Sw_n - w_n\|^2.\end{aligned}$$

Combining this inequality and (i), we directly obtain (ii). $\square$

Now we suppose that the STOP, in Step 5, never occurs in Algorithm 2, and we prove the strong convergence of the infinite sequence $\{x_n\}$ generated by this algorithm, to the projection of x_0 onto $E(f) \cap \mathrm{Fix}(S)$.

Theorem 4.1 *Let C be a nonempty, closed, and convex subset of H. Let f be a function from $\Delta \times \Delta$ into $\mathbb{R}$ satisfying conditions* (A1), (A2), (A3bis), (A4), *and let S be a mapping from C to C satisfying condition* (B1), *and such that $E(f) \cap \mathrm{Fix}(S) \neq \emptyset$. Let $\alpha \in]0,2[$, $\gamma \in]0,1[$, and suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ satisfy the conditions* (Q). *Then the sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to the projection of x_0 onto $E(f) \cap \mathrm{Fix}(S)$.*

Proof Let $\{x_n\}$ be the infinite sequence generated by Algorithm 2. Since we use the hybrid projection method, it follows from our discussion in Sect. 3 that the sequence $\{x_n\}$ generated by Algorithm 2 satisfies the inequalities (4), and thus is bounded. Consequently, from Proposition 3.3, the strong convergence of the sequence $\{x_n\}$ to the projection of x_0 onto $E(f) \cap \mathrm{Fix}(S)$ holds provided that every weak limit point of $\{x_n\}$ is an element of $E(f) \cap \mathrm{Fix}(S)$. So, let $\bar{x}$ be a weak limit point of $\{x_n\}$, and suppose that $x_{n_i} \rightharpoonup \bar{x}$. Since C is closed and convex, it is also weakly closed, and thus $\bar{x} \in C$. The proof of $\bar{x} \in E(f) \cap \mathrm{Fix}(S)$ is done in several steps.

Step 1. $\|x_{n+1} - x_n\| \to 0$, $\|x_n - t_n\| \to 0$, $\sigma_n \|g_n\| \to 0$, $\|Sw_n - w_n\| \to 0$.

The proof of the first two limits is the same as the proof of the first step of Theorem 3.1. On the other hand, let $x^* \in E(f) \cap \mathrm{Fix}(S)$. Then, from Proposition 4.4(ii), it follows that the next two inequalities

$$(1-\alpha_n)\sigma_n^2 \|g_n\|^2 \le \|x_n - t_n\| \big[\|x_n - x^*\| + \|t_n - x^*\| \big],$$

$$(1-\alpha_n)(1-\beta_n)(\beta_n - \xi)\|Sw_n - w_n\|^2 \le \|x_n - t_n\| \big[\|x_n - x^*\| + \|t_n - x^*\| \big],$$

hold for every $n \in \mathbb{N}$. Since $1-\alpha_n \ge 1-c > 0$, $1-\beta_n \ge 1-b > 0$, $\beta_n - \xi \ge d - \xi > 0$, for every $n \in \mathbb{N}$, since $\|x_n - t_n\| \to 0$, and since the sequences $\{x_n\}$ and $\{t_n\}$ are bounded, we easily deduce from the first inequality that $\sigma_n \|g_n\| \to 0$, and from the second inequality that $\|Sw_n - w_n\| \to 0$.

Step 2. The sequences $\{y_{n_i}\}$, $\{z_{n_j}\}$, and $\{g_{n_i}\}$ are bounded. Furthermore, the sequence $\{f(z_{n_i}, x_{n_i})\}$ converges to zero as $i \to \infty$.

In order to show that the sequence $\{y_{n_i}\}$ is bounded, it suffices to prove that there exists $M > 0$ such that $\|x_{n_i} - y_{n_i}\| \le M$ for i large enough. Without loss of generality, we can suppose that $x_{n_i} \neq y_{n_i}$ for all i, and we set

$$A_i(y) = \lambda_{n_i} f(x_{n_i}, y) + \frac{1}{2}\|y - x_{n_i}\|^2,$$

for every $y \in C$. Since $f(x_{n_i}, \cdot)$ is convex, it is easy to see that $A_i(y)$ is a strongly convex function on C with modulus $\nu = 1$. So, for all $y_1, y_2 \in C$, $s(y_1) \in \partial A_i(y_1)$ and $s(y_2) \in \partial A_i(y_2)$, we have the inequality

$$\big\langle s(y_1) - s(y_2), y_1 - y_2 \big\rangle \ge \|y_1 - y_2\|^2.$$

Taking $y_1 = x_{n_i}$ and $y_2 = y_{n_i}$ and noting that $0 \in \partial A_i(y_{n_i}) + N_C(y_{n_i})$ by definition of y_{n_i}, we obtain that there exists $s(y_{n_i}) \in \partial A_i(y_{n_i})$ such that $-s(y_{n_i}) \in N_C(y_{n_i})$, i.e.,

$$\big\langle -s(y_{n_i}), y - y_{n_i} \big\rangle \le 0 \quad \text{for every } y \in C. \tag{13}$$

So, for every $s(x_{n_i}) \in \partial A_i(x_{n_i})$, we have

$$\langle s(x_{n_i}), x_{n_i} - y_{n_i} \rangle \geq \langle s(y_{n_i}), x_{n_i} - y_{n_i} \rangle + \|x_{n_i} - y_{n_i}\|^2$$
$$\geq \|x_{n_i} - y_{n_i}\|^2,$$

where we have used (13) with $y = x_{n_i}$. Consequently, we obtain that

$$\|x_{n_i} - y_{n_i}\| \leq \|s(x_{n_i})\| \quad \text{for every } s(x_{n_i}) \in \partial A_i(x_{n_i}). \tag{14}$$

On the other hand, since $x_{n_i} \rightharpoonup \bar{x}$, it follows from Proposition 4.3, that for any $\varepsilon > 0$, there exist $\eta > 0$ and $i_0 \in \mathbb{N}$ such that

$$\partial_2 f(x_{n_i}, x_{n_i}) \subset \partial_2 f(\bar{x}, \bar{x}) + \frac{\varepsilon}{\eta} B \tag{15}$$

holds for all $i \geq i_0$, where B denotes the closed unit ball of H.

Since $s(x_{n_i}) \in \partial A_i(x_{n_i}) = \lambda_{n_i} \partial_2 f(x_{n_i}, x_{n_i})$ for all i, and the set $\partial_2 f(\bar{x}, \bar{x})$ is bounded, and as $0 < \lambda \leq \lambda_{n_i} \leq 1$ for all i, the inclusion (15) implies that the right-hand side of (14) be bounded. So, there exists $M > 0$ such that $\|x_{n_i} - y_{n_i}\| \leq M$ for all $i \geq i_0$, and the sequence $\{y_{n_i}\}$ is bounded.

The sequence $\{z_{n_i}\}$, being a convex combination of the sequences $\{x_{n_i}\}$ and $\{y_{n_i}\}$, is also bounded, and there exists a subsequence of $\{z_{n_i}\}$, again denoted $\{z_{n_i}\}$, that converges weakly to $\bar{z} \in C$. Then it follows from Proposition 4.3, that for any $\varepsilon > 0$, there exists $\eta > 0$ and i_0 such that

$$\partial_2 f(z_{n_i}, x_{n_i}) \subset \partial_2 f(\bar{z}, \bar{x}) + \frac{\varepsilon}{\eta} B$$

holds for all $i \geq i_0$. Since $g_{n_i} \in \partial_2 f(z_{n_i}, x_{n_i})$ for all i, and as B and $\partial_2 f(\bar{z}, \bar{x}))$ are bounded, we deduce that $\{g_{n_i}\}$ is bounded. Then,

$$f(z_{n_i}, x_{n_i}) = \sigma_{n_i} \|g_{n_i}\| \|g_{n_i}\| \to 0.$$

Indeed, the previous equality comes from the definition of σ_{n_i} when $y_{n_i} \neq x_{n_i}$, from $\sigma_{n_i} = 0$ and $f(z_{n_i}, x_{n_i}) = 0$ (because $z_{n_i} = x_{n_i}$) when $y_{n_i} = x_{n_i}$. Finally, the convergence to zero follows directly from the boundedness of $\{g_{n_i}\}$ and from $\sigma_{n_i}\|g_{n_i}\| \to 0$ (see Step 1 of this proof).

Step 3. $\|x_{n_i} - y_{n_i}\| \to 0$, and $\bar{x}$ belongs to $E(f)$.

If $y_{n_i} = x_{n_i}$ for an infinite number of indices n_i, then it follows from Proposition 3.1(i) that, for every $i \in \mathbb{N}$,

$$0 \leq \lambda_{n_i} f(x_{n_i}, y) \quad \forall y \in C. \tag{16}$$

Since $\lambda_{n_i} \in [\lambda, 1]$ with $\lambda > 0$ and since $f(\cdot, y)$ is weakly continuous on the open set $\Delta \supset C$, we obtain, after taking the limit in (16) as $i \to \infty$, that $0 \leq f(\bar{x}, y)$ for every $y \in C$, i.e., $\bar{x} \in E(f)$.

Now, we suppose that $y_{n_i} \neq x_{n_i}$ for i large enough, and let i be such an index. The function $f(z_{n_i}, \cdot)$ being convex, we can write

$$\rho_{n_i} f(z_{n_i}, y_{n_i}) + (1 - \rho_{n_i}) f(z_{n_i}, x_{n_i})$$
$$\geq f(z_{n_i}, \rho_{n_i} y_{n_i} + (1 - \rho_{n_i}) x_{n_i}) = f(z_{n_i}, z_{n_i}) = 0.$$

Hence, $\rho_{n_i}[f(z_{n_i}, x_{n_i}) - f(z_{n_i}, y_{n_i})] \leq f(z_{n_i}, x_{n_i})$. Now, by definition of the line-search, we have

$$\frac{\alpha}{2\lambda_{n_i}} \|y_{n_i} - x_{n_i}\|^2 \leq f(z_{n_i}, x_{n_i}) - f(z_{n_i}, y_{n_i}).$$

Multiplying both sides of the previous inequality by ρ_{n_i}, we obtain

$$\begin{aligned} \frac{\rho_{n_i}\alpha}{2\lambda_{n_i}} \|y_{n_i} - x_{n_i}\|^2 &\leq \rho_{n_i}\big[f(z_{n_i}, x_{n_i}) - f(z_{n_i}, y_{n_i})\big] \\ &\leq f(z_{n_i}, x_{n_i}) \to 0. \end{aligned} \tag{17}$$

Since $0 < \lambda \leq \lambda_{n_i} \leq 1$ for every i, it follows from (17) that

$$\|x_{n_i} - y_{n_i}\| \to 0 \quad \text{and} \quad y_{n_i} \rightharpoonup \bar{x}. \tag{18}$$

Indeed, in the case where $\limsup \rho_{n_i} > 0$, there exist $\bar{\rho} > 0$ and a subsequence of $\{\rho_{n_i}\}$, denoted again by $\{\rho_{n_i}\}$, such that $\rho_{n_i} \to \bar{\rho}$. Then, by (17), $\|x_{n_i} - y_{n_i}\| \to 0$ as $i \to \infty$. In the case where $\rho_{n_i} \to 0$, let $\{m_i\}$ be the sequence of the smallest positive integers such that, for every i,

$$f(z_{n_i}, x_{n_i}) - f(z_{n_i}, y_{n_i}) \geq \frac{\alpha}{2\lambda_{n_i}} \|x_{n_i} - y_{n_i}\|^2,$$

where $z_{n_i} = (1 - \gamma^{m_i})x_{n_i} + \gamma^{m_i} y_{n_i}$. Since $\rho_{n_i} = \gamma^{m_i} \to 0$, it follows that $m_i > 1$ for i sufficiently large, and consequently that

$$f(\bar{z}_{n_i}, x_{n_i}) - f(\bar{z}_{n_i}, y_{n_i}) < \frac{\alpha}{2\lambda_{n_i}} \|x_{n_i} - y_{n_i}\|^2, \tag{19}$$

where $\bar{z}_{n_i} = (1 - \gamma^{m_i - 1})x_{n_i} + \gamma^{m_i - 1} y_{n_i}$. On the other hand, using Proposition 4.4(i) with $n = n_i$ and $y = x_{n_i}$, we can write

$$\|x_{n_i} - y_{n_i}\|^2 \leq -\lambda_{n_i} f(x_{n_i}, y_{n_i}). \tag{20}$$

Combining (19) with (20), we obtain

$$f(\bar{z}_{n_i}, x_{n_i}) - f(\bar{z}_{n_i}, y_{n_i}) < -\frac{\alpha}{2} f(x_{n_i}, y_{n_i}). \tag{21}$$

Taking the limit in (21) as $i \to \infty$, using the weak continuity of f, and recalling that $x_{n_i} \rightharpoonup \bar{x}$, $y_{n_i} \rightharpoonup \bar{y}$, and $\gamma^{m_i} \to 0$, we have that $\bar{z}_{n_i} \rightharpoonup \bar{x}$ and

$$-f(\bar{x}, \bar{y}) \leq -\frac{\alpha}{2} f(\bar{x}, \bar{y}),$$

i.e., $f(\bar{x}, \bar{y}) \geq 0$, because $\alpha \in]0, 2[$. Then, taking the limit in (20) as $i \to \infty$, we have that $\|x_{n_i} - y_{n_i}\| \to 0$, and thus, since $x_{n_i} \rightharpoonup \bar{x}$, that $y_{n_i} \rightharpoonup \bar{x}$. Since $\lambda_{n_i} \in [\lambda, 1]$ with $\lambda > 0$, we obtain, after taking the limit on i in Proposition 3.1(i) with $n = n_i$, that $0 \leq f(\bar{x}, y)$ for every $y \in C$, i.e., $\bar{x} \in E(f)$.

Step 4. $\bar{x}$ belongs to $\mathrm{Fix}(S)$.

Since $\|w_n - x_n\| \leq \|x_n - \sigma_n g_n - x_n\| = \sigma_n \|g_n\| \to 0$, we have immediately that $\|w_n - x_n\| \to 0$, and thus that $w_{n_i} \rightharpoonup \bar{x}$. Then the operator $I - S$ being demiclosed at zero, and $\|Sw_{n_i} - w_{n_i}\| \to 0$, we can conclude that $S\bar{x} - \bar{x} = 0$, i.e., $\bar{x} \in \mathrm{Fix}(S)$. □

In Algorithm 2, two projections are needed to get w_n. First, x_n is projected onto the half-space

$$H_n = \{x \in H : \langle g_n, x_n - x\rangle \geq f(z_n, x_n)\}$$

to get $x_n - \sigma_n g_n$, and after that, this point is projected onto C to get w_n. In the next algorithm, denoted Algorithm 2a, only one projection of x_n is performed onto the intersection $C \cap H_n$ to get w_n. In other words, Algorithm 2a is the same as Algorithm 2, except that step 4 is replaced by

Step 4a. Select $g_n \in \partial_2 f(z_n, x_n)$ and compute $w_n = P_{C\cap H_n}(x_n)$, where $H_n = \{x \in H : \langle g_n, x_n - x\rangle \geq f(z_n, x_n)\}$.

In order to prove the strong convergence of the sequence $\{x_n\}$ generated by Algorithm 2a, it is interesting to rewrite Step 4a under an equivalent form. It is the aim of the next proposition.

Proposition 4.5 *Let $w_n = P_{C\cap H_n}(x_n)$. Then $w_n = P_{C\cap H_n}(u_n)$ where $u_n = P_{H_n}(x_n)$.*

Proof First observe that when $y_n = x_n$, we have that $z_n = x_n$ and $x_n \in H_n$. Consequently, in that case, $u_n = x_n$ and $w_n = P_{C\cap H_n}(u_n)$. Next, we consider the case when $y_n \neq x_n$ for every n. Taking any $y \in C \cap H_n$, we have to prove that $\|w_n - u_n\| \leq \|y - u_n\|$. Since $f(x, .)$ is convex, we have

$$t_n f(z_n, y_n) + (1 - t_n) f(z_n, x_n) \geq f(z_n, z_n) = 0,$$

where $0 < t_n < 1$. Using the linesearch, we deduce from the previous inequality that

$$f(z_n, x_n) \geq t_n\big[f(z_n, x_n) - f(z_n, y_n)\big] \geq \frac{t_n\alpha}{2\lambda_n}\|y_n - x_n\|^2 > 0.$$

But this implies that $x_n \notin H_n$. Hence, there exists $\mu \in [0, 1[$ such that

$$\tilde{x} \equiv \mu x_n + (1 - \mu) y \in C \cap \mathrm{bd}(H_n),$$

where $\mathrm{bd}(H_n) = \{x \in H : \langle g_n, x_n - x\rangle = f(x_n, z_n)\}$. Using the definition of u_n, we obtain that $\langle \tilde{x} - u_n, x_n - u_n\rangle \leq 0$. Consequently, we can write successively

$$\begin{aligned} \|y - u_n\|^2 &\geq (1-\mu)^2\|y - u_n\|^2 = \big\|\tilde{x} - \mu x_n - (1-\mu)u_n\big\|^2 \\ &= \big\|\tilde{x} - u_n - \mu(x_n - u_n)\big\|^2 \\ &= \|\tilde{x} - u_n\|^2 + \mu^2\|x_n - u_n\|^2 - 2\mu\langle \tilde{x} - u_n, x_n - u_n\rangle \\ &\geq \|\tilde{x} - u_n\|^2 + \mu^2\|x_n - u_n\|^2 \\ &\geq \|\tilde{x} - u_n\|^2. \end{aligned} \tag{22}$$

Furthermore, since $x_n \notin H_n$, we have that $u_n = P_{\mathrm{bd}(H_n)}(x_n)$ and $w_n = P_{C\cap \mathrm{bd}(H_n)}(x_n)$. Then, using successively the Pythagoras theorem, the definitions of w_n and u_n, we obtain

$$\begin{aligned}\|\tilde{x}-u_n\|^2 &= \|\tilde{x}-x_n\|^2 - \|u_n-x_n\|^2 \\ &\geq \|w_n-x_n\|^2 - \|u_n-x_n\|^2 \\ &\geq \|w_n-u_n\|^2. \end{aligned} \tag{23}$$

From (22) and (23), we obtain that $\|y-u_n\|^2 \geq \|w_n-u_n\|^2$ for every $y \in C \cap H_n$, i.e., $w_n = P_{C\cap H_n}(u_n)$. □

Thanks to this proposition, the only difference between the two algorithms is in the computation of w_n. For Algorithm 2, we have $w_n = P_C(u_n)$, while for Algorithm 2a, we have $w_n = P_{C\cap H_n}(u_n)$. Since $E(f) \cap \mathrm{Fix}(S) \subset C \cap H_n \subset C$ (see the proof of Theorem 4.2), we have that the point w_n, computed in Algorithm 2a, is closer to the solution set than the point w_n computed in Algorithm 2.

Theorem 4.2 *Let C be a nonempty, closed and convex subset of H. Let f be a function from $\Delta \times \Delta$ into $\mathbb{R}$ satisfying conditions* (A1), (A2), (A3bis), (A4), *and S be a mapping from C to C satisfying condition* (B1), *and such that $E(f) \cap \mathrm{Fix}(S) \neq \emptyset$. Let $\alpha \in]0,2[$, $\gamma \in]0,1[$, and suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\lambda_n\}$ satisfy the conditions* (Q): *Then the sequence $\{x_n\}$ generated by Algorithm 2a converges strongly to the projection of x_0 onto $E(f) \cap \mathrm{Fix}(S)$.*

Proof Let $x^* \in \mathrm{Fix}(S) \cap E(f)$. First, we observe that x^* and z_n belong to $C \cap H_n$, for every integer $n \geq 1$. Indeed, by definition of g_n, we have

$$f(z_n, y) \geq f(z_n, x_n) + \langle g_n, y - x_n\rangle \quad \forall y \in C.$$

Taking $y = x^*$, and remembering that $f(z_n, x^*) \leq 0$, we deduce that $\langle g_n, x_n - x^*\rangle \geq f(z_n, x_n)$, i.e., $x^* \in H_n$. Similarly, taking $y = z_n$, and remembering that $f(z_n, z_n) = 0$, we obtain that $\langle g_n, x_n - z_n\rangle \geq f(z_n, x_n)$, i.e., $z_n \in H_n$.

The rest of the proof of this theorem is similar to the proof of Theorem 4.1, except the first part of step 4, namely, the proof of $w_{n_i} \rightharpoonup \bar{x}$. In that purpose, let $x_{n_i} \rightharpoonup \bar{x}$. Since $z_{n_i} \in C \cap H_{n_i}$, $w_{n_i} = P_{C\cap H_{n_i}}(u_{n_i})$, and $u_{n_i} = P_{H_{n_i}}(x_{n_i})$, we have

$$\|w_{n_i} - z_{n_i}\| \leq \|u_{n_i} - z_{n_i}\| \leq \|x_{n_i} - z_{n_i}\|.$$

Hence, $\|w_{n_i} - z_{n_i}\| \to 0$ because $\|x_{n_i} - z_{n_i}\| = \gamma_{n_i}\|x_{n_i} - y_{n_i}\| \to 0$ (see Step 3 of the proof of Theorem 4.1). Consequently, it follows from $x_{n_i} \rightharpoonup \bar{x}$ that $z_{n_i} \rightharpoonup \bar{x}$, and thus that $w_{n_i} \rightharpoonup \bar{x}$. □

5 The Particular Case of Variational Inequalities

In this section, we consider the particular Ky Fan inequality corresponding to the function f defined, for every $x, y \in C$, by

$$f(x,y) = \big\langle F(x), y - x\big\rangle, \quad \text{with } F : C \to H.$$

Doing so, we obtain the classical variational inequality: Find $x^* \in C$ such that

$$\big\langle F\big(x^*\big), y - x^*\big\rangle \geq 0 \quad \text{for every } y \in C.$$

The set of solutions of this problem is denoted by VI(F). In that particular situation, the solution y_n of the minimization problem

$$\min_{y\in C}\left\{\lambda_n f(x_n, y) + \frac{1}{2}\|y - x_n\|^2\right\}$$

can be expressed as $y_n = P_C(x_n - \lambda_n F(x_n))$. Furthermore, the Armijo condition

$$f(z_n, x_n) - f(z_n, y_n) \geq \frac{\alpha}{2\lambda_n}\|x_n - y_n\|^2$$

gives rise to the inequality

$$\left\langle F(z_n), x_n - y_n\right\rangle \geq \frac{\alpha}{2\lambda_n}\|x_n - y_n\|^2.$$

Then it is easy to see that Algorithm 2 can be rewritten as follows.

Algorithm 2-VI

Step 0. Choose $\alpha \in]0, 2[$, $\gamma \in]0, 1[$ and the sequences

$$\{\alpha_n\} \subset [0, 1[, \qquad \{\beta_n\} \subset]0, 1[\quad \text{and} \quad \{\lambda_n\} \subset]0, 1].$$

Step 1. Let $x_0 \in C$. Set $n = 0$.
Step 2. Compute $y_n = P_C(x_n - \lambda_n F(x_n))$.
Step 3. If $y_n = x_n$, then set $z_n = x_n$. Otherwise

Step 3.1. Find m the smallest nonnegative integer such that

$$\begin{cases} \langle F(z_{n,m}), x_n - y_n\rangle \geq \frac{\alpha}{2\lambda_n}\|x_n - y_n\|^2 \text{ where} \\ z_{n,m} = (1-\gamma^m)x_n + \gamma^m y_n. \end{cases}$$

Step 3.2. Set $\rho_n = \gamma^m$, $z_n = z_{n,m}$, and go to Step 4.

Step 4. Compute $w_n = P_C(x_n - \sigma_n F(z_n))$ where $\sigma_n = \frac{\langle F(z_n), x_n - z_n\rangle}{\|F(z_n)\|^2}$ if $y_n \neq x_n$, and $\sigma_n = 0$ otherwise.
Step 5. Compute $t_n = \alpha_n x_n + (1-\alpha_n)[\beta_n w_n + (1-\beta_n)S w_n]$.
If $y_n = x_n$ and $t_n = x_n$, then STOP: $x_n \in \mathrm{VI}(F) \cap \mathrm{Fix}(S)$.
Otherwise, go to Step 6.
Step 6. Compute $x_{n+1} = P_{C_n \cap D_n} x_0$, where

$$C_n = \left\{z \in C : \|t_n - z\| \leq \|x_n - z\|\right\} \quad \text{and}$$

$$D_n = \left\{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\right\}.$$

Step 7. Set $n := n + 1$, and go to step 2.

In order to apply Theorem 4.1 to the variational inequality, let us observe that, in this particular case, conditions (A1) and (A4) are always satisfied, and that condition (A2) becomes: F is pseudomonotone on C, i.e.,

$$\left\langle F(x), y - x\right\rangle \geq 0 \quad \Rightarrow \quad \left\langle F(y), x - y\right\rangle \leq 0 \quad \text{for every } x, y \in C.$$

Furthermore, if $F : \Delta \to H$ is such that, for any sequence $\{x_n\} \subset \Delta$,

$$x_n \rightharpoonup x \quad \Rightarrow \quad F(x_n) \to F(x), \tag{24}$$

then the corresponding function f is weakly continuous on $\Delta \times \Delta$.

Taking account of these properties, Theorem 4.1 can be rewritten as the following.

Theorem 5.1 *Let C be a nonempty, closed, and convex subset of H. Let $F : \Delta \to H$ be a mapping, pseudomonotone on C, and satisfying* (24). *Let S be a mapping from C to C satisfying conditions* (B1), *and such that* $\mathrm{VI}(F) \cap \mathrm{Fix}(S) \neq \emptyset$. *Let* $\alpha \in]0, 2[$, $\gamma \in]0, 1[$, *and suppose that the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\lambda_n\}$ *satisfy the conditions* (Q). *Then the sequence $\{x_n\}$ generated by Algorithm* 2-VI *converges strongly to the projection of x_0 onto* $\mathrm{VI}(F) \cap \mathrm{Fix}(S)$.

Finally, to obtain the algorithm and the convergence theorem corresponding to Algorithm 2a, but for the variational inequality, it suffices to replace in step 4 the calculation of w_n by $w_n = P_{C\cap H_n}(x_n)$ where

$$H_n = \bigl\{x \in H : \bigl\langle F(z_n), z_n - x\bigr\rangle \geq 0\bigr\}.$$

6 Conclusion

Two new iterative methods have been introduced for finding a common element of the set of points satisfying a Ky Fan inequality and the set of fixed points of a contraction mapping in a Hilbert space. The basic iteration used in this paper is the extragradient iteration with or without the inclusion of a linesearch procedure. The strong convergence of the iterates has been obtained thanks to a hybrid projection method. Another approach, known as the viscosity method, has been studied in the literature to obtain strong convergence theorems. This approach could be an interesting variant to the method developed in this paper. This will be the subject of a forthcoming research paper.

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